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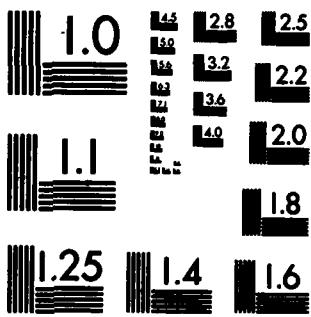
RISK-EFFICIENT ESTIMATION OF THE MEAN EXPONENTIAL
SURVIVAL TIME UNDER RAN. (U) MICHIGAN STATE UNIV EAST
LANSING DEPT OF STATISTICS AND PROBA.
J C GARDINER ET AL. JAN 83 NO0014-79-C-0522 F/G 12/1

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Risk-efficient estimation of the mean exponential survival time under random censoring

Joseph C. Gardiner*
Michigan State University, East Lansing, Michigan 48824

V. Susarla[†]
State University of New York, Binghamton, New York 13901

Abstract: The paper proposes a sequential estimator $\hat{\theta}$ of the parameter θ of an exponential distribution when the data is censored. Without any further conditions, it is shown that $\hat{\theta}$ is asymptotically risk efficient when the loss is measured by the squared error loss of estimation of θ plus a linear cost function of the number of observations. In addition, it is shown that $\hat{\theta}$ is asymptotically normal as the cost per observation goes to zero.

Key Words and Phrases: censored data, exponential distribution, and sequential estimation.

*Research supported in part by the National Institutes of Health under grant 1R01-GM28030 and by the Office of Naval Research under contract N00014-79-C0522.

[†]Research supported in part by the National Institutes of Health under grant 1R01-GM28405.

1. Introduction.

In several longitudinal investigations the estimation of the mean survival time is of basic importance. This is usually based on the data gathered from a sample of $n (\geq 1)$ identical units on test as in a clinical trial or life test. However, it is often the case in these survival studies that the lifetimes of the specimens are not completely observable due to random withdrawals and consequent loss to follow-up. Accordingly, for each survival time X we envisage a competing censoring time Y , but the only datum available is (Z, δ) where $Z = \min(X, Y)$ and $\delta = 1$ whenever $X \leq Y$ and $\delta = 0$ otherwise. We shall assume X to be independent of Y and $Z \geq 0$ almost surely (a.s.) for each θ .

Suppose $\{(Z_i, \delta_i) : 1 \leq i \leq n\}$ is a random sample of size n with lifetimes X_i having common exponential survival function $F(t) = \exp(-t/\theta)$, $t > 0$ with $\theta (> 0)$ unknown and the censoring times Y_i having common (unknown) survival function G . We consider the sequence of estimators $\{\hat{\theta}_n\}$ of θ given by

$$\hat{\theta}_n = \begin{cases} (\sum_{i=1}^n Z_i) / (\sum_{i=1}^n \delta_i) & \text{if } \sum_{i=1}^n \delta_i \geq 1 \\ 0 & \text{if } \sum_{i=1}^n \delta_i = 0. \end{cases} \quad [1.1]$$

$\hat{\theta}_n$ is easily seen to converge a.s. to θ as $n \rightarrow \infty$ for each θ .

The loss incurred in estimation of θ by $\hat{\theta}_n$ is

$$L_n(a, c) = a(\hat{\theta}_n - \theta)^2 + cn \quad [1.2]$$

where $a (> 0)$ is a known constant and $c (> 0)$ is the cost per unit observation. With the definition [1.1] we can show that

$$E(\hat{\theta}_n - \theta)^2 = (a\theta^2/E\delta)n^{-1} + o(n^{-1}) \quad [1.3]$$

whence the risk associated with the estimation scheme becomes

$$R_n(a, c) = EL_n(a, c) = (a\theta^2/E\delta)n^{-1} + cn + o(n^{-1}).$$

Therefore, in order to minimize this risk (with respect to n) we must

take a sample of size n_c^0 with subsequent minimum risk R_c^0 where (for small c)

$$R_c^0 \sim 2cn_c^0 \text{ and } n_c^0 \sim (a\theta^2/cE\delta)^{1/2}. \quad [1.4]$$

Since θ is the unknown parameter which we wish to estimate, this minimum risk R_c^0 and optimal sample size n_c^0 are also unknown and thus we are led naturally to explore a sequential scheme for estimating θ . In this article we propose one such procedure in which the sample size is given by a random stopping number $N(c)$ and the associated risk $R_c^* = E\delta_{N(c)}$ satisfies $R_c^*/R_c^0 \rightarrow 1$ as $c \rightarrow 0$, that is if our scheme is asymptotically risk-efficient.

Sequential procedures analogous to the one outlined here have been considered, in the absence of censoring, by several researchers beginning with the pioneering work of Robbins (1959) for the estimation of the mean of a normal population. This was later extended by Starr (1966) and Starr and Woodroffe (1969). The study of sequential point estimation of the exponential mean is taken up in Starr and Woodroffe (1972). Gardiner and Susarla (1982) make the first examination of the problem of sequential estimation of the mean survival time in the presence of censoring when both the underlying survival time and censoring distributions are unspecified. When censoring is absent, similar procedures for estimation of functionals of an unspecified distribution are discussed in Sen and Ghosh (1981), Ghosh and Mukhopadhyay (1978) with some extensions and refinements by Chow et. al. (1981, 1982). Throughout the rest of the paper, the terms involving the random variable $[\sum_{i=1}^n \delta_i = 0]$ are left out without any further indication since $P[\sum_{i=1}^n \delta_i = 0]$ goes to zero at an exponential rate and all our scale factors will be like n^{-a} with $a > 0$.

2. The Sequential Procedure

Let $m (\geq 1)$ be the initial sample size. Then motivated by [1.4] define the stopping number $N_c (\equiv N(c))$ by

$$N_c = \min\{n \geq m : n \geq \left(\frac{a}{c}\right)^{\frac{1}{2}}(\hat{\sigma}_n^{-2} + n^{-\gamma})\} \quad [2.1]$$

where $\gamma > 0$ is a constant and $\hat{\sigma}_n^2$ is an appropriate estimator of $\sigma^2(\theta) \equiv \theta^2/E\delta = \theta^3(\int_0^\infty FG)^{-1}$. Notice that by the ordinary Central Limit Theorem we have from [1.1]

$$n^{\frac{1}{2}}(\hat{\theta}_n - \theta) \xrightarrow{D} N(0, \sigma^2(\theta)). \quad [2.2]$$

An appropriate estimator of $\sigma^2(\theta)$ based on $\{(Z_i, \delta_i) : 1 \leq i \leq n\}$ is $\hat{\sigma}_n^2$ where

$$\hat{\sigma}_n^2 = \hat{\theta}_n^3/\bar{Z}_n = \{\bar{Z}_n^2/\hat{\delta}_n^3\}[\hat{\delta}_n \neq 0] \quad [2.3]$$

with the overscore denoting the usual corresponding sample mean and $[A]$ denoting the indicator of the set A . Observe that for each fixed θ , $\hat{\sigma}_n^2$ converges a.s. to $\sigma^2(\theta)$ as $n \rightarrow \infty$.

Our proposed sequential scheme will take $N(c)$ observations and estimate θ by $\hat{\theta}_{N(c)}$ with resulting risk

$$R_c^* = a E(\hat{\theta}_{N(c)} - \theta)^2 + c E(N(c)). \quad [2.4]$$

The main result of this paper is

Theorem 1. (Risk Efficiency). For each $\gamma > 0$,

$$R_c^*/R_c^0 \rightarrow 1 \text{ as } c \rightarrow 0.$$

Furthermore, we can easily demonstrate

Theorem 2. (Asymptotic Normality of $\hat{\theta}_{N(c)}$). As $c \rightarrow 0$, for each θ $\{\sqrt{N(c)}(\hat{\theta}_{N(c)} - \theta)/\hat{\sigma}_{N(c)}\}$ converges in distribution to a standard normal random variable.

The proofs of these theorems are outlined in the next section. We begin with a demonstration of the expansion [1.3] and a lemma concerning the rate of growth of $N(c)$ as $c \rightarrow 0$.

3. Proofs.

We begin with a Lemma on left truncated inverse binomial moments.

Lemma 1. Let V be a binomial random variable with parameters n, p .

Then for any positive integers k, ℓ

$$E(V^{-k} | V \geq \ell) \leq K(np)^{-k}$$

where the constant K does not depend on n and p .

Proof. Write $n^{[k]} = n(n-1)\dots(n-k+1)$ and $n^{-[k]} = (n^{[k]})^{-1}$. Then

$$\begin{aligned} & \sum_{x=0}^n \{(x+k)^{-[k]} n^{[x]} p^x (1-p)^{n-x}/x!\} \\ &= (n+k)^{-[k]} \sum_{x=0}^n \{(n+k)^{[x+k]} p^x (1-p)^{n-x}/(x+k)!\} \\ &\leq (n+k)^{-[k]} p^{-k} \end{aligned}$$

Therefore since $(x+k)^{[k]}/x^k$ is decreasing with respect to x ,

$$\begin{aligned} E(V^{-k} | V \geq \ell) &\leq (\ell+k)^{[k]} \ell^{-k} E((V+k)^{-[k]} | V \geq \ell) \\ &\leq (\ell+k)^{[k]} \ell^{-k} (n+k)^{-[k]} p^{-k} \\ &\leq K(np)^{-k}. \end{aligned}$$

We now turn to the crucial expansion given in [1.3]. To this end note that from [1.1] we can write

$$\hat{\theta}_n - \theta = U_n V_n \quad [3.1]$$

where

$$U_n = (\bar{Z}_n - \theta \bar{\delta}_n); \quad V_n = \bar{\delta}_n^{-1} \quad [\bar{\delta}_n \neq 0], \quad [3.2]$$

We then have

Lemma 2. For any $p \geq 1$, $E|U_n|^{2p} = o(n^{-p})$ and $E|V_n|^p = o(1)$.

Proof. Since δ is either 0 or 1 and $0 \leq Z \leq X$ a.s., all moments of δ and Z are finite. From [3.2], we see that U_n is the average of n iidrv's with zero mean and variance $\theta^2 E\delta(-\theta E\delta)$. Then by the Marcinkiewicz-Zygmund-Hölder inequality we obtain

$$E|U_n|^{2p} \leq k_1 n^{-p} E|Z - \theta\delta|^{2p} = O(n^{-p}).$$

(Here, and throughout this sequel k_j is a generic constant not depending on n).

For the second part of this lemma we apply Lemma 1 to the binomial random variable $n\bar{\delta}_n$, from which the required statement follows immediately. This concludes the proof of Lemma 2.

Before we proceed further, we point out that [1.3] follows from the equality $\hat{\theta}_n - \theta = (E\delta)^{-1} U_n \{1 - v_n(\bar{\delta}_n - E\delta)\}$ where U_n and v_n are defined by [3.2].

The next Lemma gives a convergence rate (as $c \downarrow 0$) for the stopping number N_c of [2.1]. First note that $N_c \geq (\frac{a}{c})^{1/2(1+\gamma)}$ a.s. so that $N_c \uparrow \infty$ a.s. as $c \downarrow 0$. In the sequel $\text{int}[x]$ denotes the greatest integer $\leq x$ and k_1, k_2, \dots denote constants independent of c , but could depend on G and θ .

Lemma 3. Let $0 < \epsilon < 1$ be arbitrary. Then as $c \downarrow 0$

$$P[N_c \leq \text{int}[n_c^0(1 - \epsilon)]] = 0(c)$$

and

$$P[N_c > \text{int}[n_c^0(1 + \epsilon)]] = 0(c).$$

[3.3]

Proof. Write $b = (\frac{a}{c})^{\frac{1}{2}}$ and $n_{1c} = \text{int}[b^{1/c(1+\gamma)}]$, $n_{2c} = \text{int}[n_c^0(1 - \epsilon)]$.

On the set $\{N_c \leq n_{2c}\}$ we have that $n \geq b\hat{\theta}_n$ for some $n \in \{n_{1c}, \dots, n_{2c}\}$. Therefore

$$\begin{aligned}
P[N_c \leq n_{2c}] &\leq P\left[\bigcup_{n=n_{1c}}^{n_{2c}} [\hat{\sigma}_n^2 \leq b n^{-1}]\right] \\
&\leq P\left[\bigcup_{n=n_{1c}}^{n_{2c}} [\hat{\sigma}_n^2 - \sigma^2 \leq b^{-2} n_{2c}^2 - \sigma^2]\right], \text{ where } \sigma^2 = \sigma^2(\theta) = \theta^2/E\delta, \\
&\leq P\left[\bigcup_{n=n_{1c}}^{n_{2c}} [|\hat{\sigma}_n^2 - \sigma^2| \geq \sigma^2 \epsilon(2 - \epsilon)]\right] \\
&= P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\hat{\sigma}_n^2 - \sigma^2| \geq \eta\right], \text{ for some } \eta > 0. \quad [3.4]
\end{aligned}$$

$$\text{Now } \hat{\sigma}_n^2 - \sigma^2 = a_n(\bar{Z}_n - EZ) - b_n(\bar{\delta}_n - E\delta) \quad [3.5]$$

where

$$a_n = (\bar{Z}_n + EZ)\bar{\delta}_n^{-3} [\bar{\delta}_n \neq 0] \text{ and}$$

$$b_n = (EZ)^2(E\delta)^{-3}(\bar{\delta}_n^{-2} + \bar{\delta}_n E\delta + (E\delta)^2)\bar{\delta}_n^{-3} [\bar{\delta}_n \neq 0].$$

Since $a_n \rightarrow 2\theta(E\delta)^{-2} = a_0$ a.s. and $b_n \rightarrow 3\theta(E\delta)^{-2} = b_0$ a.s. as $n \rightarrow \infty$,

we obtain from [3.5] that the righthand side of [3.4] may be bounded by

$$P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\bar{Z}_n - EZ| > \eta\right] + P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\bar{\delta}_n - E\delta| > \eta\right] \quad [3.6]$$

for an $\eta > 0$. Each of the two terms in [3.6] are handled in the same

fashion. Observe that $\{|\bar{Z}_n - EZ| : n_{1c} \leq n \leq n_{2c}\}$ is a reverse submartingale to which the Kolmogorov maximal inequality applies. Thus

$$P\left[\max_{n_{1c} \leq n \leq n_{2c}} |\bar{Z}_n - EZ| \geq \eta\right] \leq k_2 E|\bar{Z}_{n_{1c}} - EZ|^{4+4\gamma} = o(c),$$

where k_1 is a constant depending only on γ . Thus combining [3.4] through [3.6] yields the first statement of Lemma 3. The second is proven in much the same way and so the proof of the lemma is terminated.

We may utilize the expansion in [3.5] to obtain the uniform integrability of $\{N_c/n_c^0 : 0 < c \leq c_0\}$ for some c_0 sufficiently small; from this $E(N_c/n_c^0) \rightarrow 1$ as $c \rightarrow 0$ will follow.

To this end note from [2.1] that

$$\hat{\sigma}_{N(c)}/\sigma \leq N_c/n_c^0 \leq \hat{\sigma}_{N(c)-1}/\sigma + (N(c) - 1)^{-\gamma}/\sigma + (b\sigma)^{-1}.$$

Then since $N_c \uparrow \infty$ a.s. (as $c \uparrow 0$), we obtain $N_c/n_c^0 \rightarrow 1$ a.s.,

Furthermore

$$(N_c/n_c^0)^2 \leq 8\{(\hat{\sigma}_{N(c)-1}/\sigma)^2 + \sigma^{-2}(N(c) - 1)^{-2\gamma} + (b\sigma)^{-2}\}.$$

Thus in order to show $E(\sup_{c \leq c_0} (N_c/n_c^0)^2) < \infty$ for some c_0 sufficiently

small, we need only examine $E(\sup_{n \geq m} |\hat{\sigma}_n^2/\sigma^2 - 1|)$ for sufficiently large m .

However, from [3.5] and the convergences $a_n \rightarrow a_0$, $b_n \rightarrow b_0$ a.s. it

suffices to establish the finiteness of $E(\sup_{n \geq m} |\bar{Z}_n - EZ|)$ and

$E(\sup_{n \geq m} |\bar{\delta}_n - E\delta|)$. The second is trivial, while the first is bounded by four times

$E|\bar{Z}_m - EZ|^2$ since $\{|\bar{Z}_n - EZ| : n \geq m\}$ is a reverse submartingale.

Proof of Theorem 2. Recall [3.1] and [3.2]. From the central limit theorem

$\frac{1}{n} U_n \xrightarrow{D} N(0, \theta^2 E\delta)$, for each $\theta > 0$. Also $V_n \rightarrow (E\delta)^{-1}$ a.s. Hence

$\frac{1}{n} (\hat{\theta}_n - \theta) \xrightarrow{D} N(0, \sigma^2)$. We have noted that $\hat{\sigma}_n^2 \rightarrow \sigma^2$ a.s. (as $n \rightarrow \infty$)

and $N_c \uparrow \infty$ as $c \uparrow 0$. Hence, by [3.8], [3.13], and [3.15] below, we get

$$\sqrt{N(c)} (\hat{\theta}_{N(c)} - \theta) / \hat{\sigma}_{N(c)} \xrightarrow{D} N(0, 1) \text{ as } c \uparrow 0.$$

Proof of Theorem 1. In view of the fact that $E\{N_c/n_c^0\} \rightarrow 1$, [1.4] and

[2.4] we must verify that

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E(\hat{\theta}_{N(c)} - \theta)^2 = 1. \quad [3.7]$$

Recall the expansion [3.3]. We first show

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{(\hat{\theta}_{N(c)} - \theta)^2 [N_c \leq n_{2c}]\} = 0. \quad [3.8]$$

Now

$$E\{(\hat{\theta}_{N(c)} - \theta)^2 [N_c \leq n_{2c}]\} \leq k_3 \{EU_{N(c)}^2 [N_c \leq n_{2c}] + EW_{N(c)}^2 [N_c \leq n_{2c}]\}, \quad [3.9]$$

where

$$W_n = U_n V_n (\bar{\delta}_n - E\delta)$$

Now applying the maximal inequality to the reverse submartingale

$\{|\bar{Z}_n - \theta\bar{\delta}_n| : n_{1c} \leq n \leq n_{2c}\}$ in the third step and Lemma 3 we have

$$\begin{aligned} E\{U_{N(c)}^2 [N_c \leq n_{2c}]\} &\leq E\left(\max_{n_{1c} \leq n \leq n_{2c}} U_n^2 [N_c \leq n_{2c}]\right) \\ &\leq E^{1/(1+\gamma)} \left(\max_{n_{1c} \leq n \leq n_{2c}} |U_n|^{2+2\gamma} (P[N_c \leq n_{2c}])^{\gamma/(1+\gamma)}\right) \\ &\leq k_4 E^{1/(1+\gamma)} |U_{n_{1c}}|^{2+2\gamma} 0(c^{\gamma/(1+\gamma)}) \\ &= 0(n_{1c}^{-1}) 0(c^{\gamma/(1+\gamma)}). \end{aligned}$$

Since $n_{1c} \sim c^{-1/2(1+\gamma)}$ and $(cn_c^0)^{-1} \sim c^{-\frac{1}{2}}$ we have that

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{U_{N(c)}^2 [N_c \leq n_{2c}]\} = 0. \quad [3.10]$$

$$\begin{aligned} \text{Similarly } E\{W_{N(c)}^2 [N_c \leq n_{2c}]\} &= \sum_{n=n_{1c}}^{n_{2c}} E\{W_n^2 [N_c = n]\} \\ &\leq \left(\sum_{n=n_{1c}}^{n_{2c}} E W_n^{2p}\right)^{1/p} (P[N_c \leq n_{2c}])^{1-1/p}, \quad p > 1. \end{aligned} \quad [3.11]$$

$$\begin{aligned} \text{Now } E W_n^{2p} &= E\{U_n^{2p} V_n^{2p} (\bar{\delta}_n - E\delta)^{2p}\} \\ &\leq E^{1/3} \{U_n^{6p}\}^{1/3} \{V_n^{6p}\}^{1/3} (\bar{\delta}_n - E\delta)^{6p} \\ &= \{0(n^{-3p}) 0(1) 0(n^{-3p})\}^{1/3} = 0(n^{-2p}), \end{aligned}$$

Thus the rhs in [3.11] is $0(n_{1c}^{-2+1/p} c^{1-1/p}) = 0(c^h)$, where

$h = \{2(1 + \gamma)\}^{-1}(2 - 1/p) + (1 - 1/p) > \frac{1}{2}$ whenever $p > (2\gamma + 3)/(3 + \gamma)$. So

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{W_{N(c)}^2 [N_c \leq n_{2c}]\} = 0 \quad [3.12]$$

Collecting our results [3.10] and [3.12] we obtain [3.8]. An entirely analogous argument using the second part of Lemma 3 will show

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{\hat{\theta}_{N(c)} - \theta\}^2 [N_c > n_{3c}] = 0, \quad [3.13]$$

where $n_{3c} = [n_c^0(1 + \epsilon)]$. Therefore, in order to verify [3.7] we are

left with proving

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{(\hat{\theta}_{N(c)} - \theta)^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.14]$$

However, from [1.3] $\lim_{c \downarrow 0} (cn_c^0)^{-1} E(\hat{\theta}_{n_c^0} - \theta)^2 = 1$, and further from the inequality

$$\begin{aligned} & E\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})(\hat{\theta}_{n_c^0} - \theta) [n_{2c} < N_c \leq n_{3c}]\} \\ & \leq E^{\frac{1}{2}}\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} E^{\frac{1}{2}}(\hat{\theta}_{n_c^0} - \theta)^2, \end{aligned}$$

we see that it suffices to establish

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.15]$$

Once again we utilize the expression [3.3] from which we get

$$\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0} = (E\delta)^{-1} \{U_{N(c)} - U_{n_c^0}\} - (W_{N(c)} - W_{n_c^0}\}. \quad [3.16]$$

Observe that

$$\begin{aligned} & E\{(\hat{\theta}_{N(c)} - \hat{\theta}_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} \\ & \leq k_5 [E\{(U_{n_c^0} - U_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\}] \\ & \quad + E\{W_{N(c)} - W_{n_c^0}\}^2 [n_{2c} < N_c \leq n_{3c}] \} \end{aligned} \quad [3.17]$$

Now

$$E\{(U_{n_c^0} - U_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} \quad [3.18]$$

$$\leq \max [E\{ \max_{n_{2c} < n \leq n_c^0} (U_n - U_{n_c^0})^2 \}, E\{ \max_{n_c^0 < n \leq n_{3c}} (U_n - U_{n_c^0})^2 \}].$$

Since $\{U_n : n_{2c} < n \leq n_c^0\}$ is a reverse martingale, $\{(U_n - U_{n_c^0})^2 : n_{2c} < n \leq n_c^0\}$

is a reverse submartingale to which the maximal inequality will be applied.

This yields

$$\begin{aligned} E\left\{\max_{n_{2c} < n \leq n_c^0} (U_n - U_{n_c^0})^2\right\} &\leq 4 E\{U_{n_{2c}} - U_{n_c^0}\}^2 \\ &= 4 \{EU_{n_{2c}}^2 - EU_{n_c^0}^2\} \end{aligned} \quad [3.19]$$

Now $EU_n^2 = (\theta^2 E\delta)n^{-1}$, $n_{2c} = \text{int}[n_c^0(1 - \epsilon)]$ and $cn_c^0 = O(1)$. Therefore the rhs of [3.19] is of order $(cn_c^0)\epsilon(1 - \epsilon)^{-1}O(1)$. Hence, by choosing ϵ arbitrarily small we have that

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\left\{\max_{n_{2c} < n \leq n_c^0} (U_n - U_{n_c^0})^2\right\} = 0. \quad [3.20]$$

The second term in [3.18] is handled in an entirely analogous fashion and will yield the result paralleling [3.20]. So

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{(U_{N(c)} - U_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.21]$$

Finally we must dispose of the second term in [3.17]. This is easily handled using the bound $EW_n^2 = O(n^{-2})$ and the simple inequalities

$$\begin{aligned} &E\{(W_{N(c)} - W_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} \\ &\leq 4 \{EW_{n_c^0}^2 + \sum_{n=n_{2c}}^{n_{3c}} EW_n^2\} = O(n_c^{0-2}) + O(n_c^{0-1})\epsilon(1 - \epsilon)^{-2}. \end{aligned}$$

Then on selecting ϵ arbitrarily small we obtain

$$\lim_{c \downarrow 0} (cn_c^0)^{-1} E\{(W_{N(c)} - W_{n_c^0})^2 [n_{2c} < N_c \leq n_{3c}]\} = 0. \quad [3.22]$$

Therefore in conjunction with [3.17], [3.21], and [3.22] yield [3.15].

This concludes the proof of Theorem 1.

4. Concluding Remarks.

To start with, it is worth pointing out that the results are true even if G is degenerate at any point T . That is, the results hold even in the fixed point truncation of X_1 . Although the results are stated here only when Y_1, \dots, Y_n, \dots are i.i.d., similar results hold even if Y_1, \dots, Y_n, \dots are assumed to be independent only. In the latter case however, some sort of condition like $\lim_{n \rightarrow \infty} \int_0^\infty FG_n > 0$, with $n \bar{G}_n = \sum_{i=1}^n G_i$, needs to be imposed.

The final comment concerns the damping factor $n^{-\gamma}$ introduced in the definition (2.1) of the stopping random variable N_c . Without this factor, the sequential estimator $\hat{\theta}_{N_c}$ may not be asymptotically risk efficient as the following argument suggests. Define

$$N_{c,1} = \min\{n \geq m: n \geq \left(\frac{a}{c}\right)^{\frac{1}{2}} \hat{\sigma}_n\} \quad [4.1]$$

where $\hat{\sigma}_n$ is defined by [2.3]. With these definitions, it can be seen that if the censoring distribution G has positive mass p at zero, then the risk $R_{c,1}^*$ of the sequential procedure $\hat{\theta}_{N_{c,1}}$ is at least $\theta^2 p^m$ and $R_{c,1}^*/R_c^0 \rightarrow \infty$ as $c \downarrow 0$ and hence $\hat{\theta}_{N_{c,1}}$ is asymptotically totally risk inefficient. Thus the factor $n^{-\gamma}$ in the definition of N_c appears to be important.

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1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Risk-efficient estimation of the mean exponential survival time under random censoring		5. TYPE OF REPORT & PERIOD COVERED
7. AUTHOR(s) Joseph C. Gardiner V. Susarla		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Michigan State University East Lansing, MI 48824		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE January 1983
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Statistics & Probability Program Office of Naval Research Arlington, VA 22217		13. NUMBER OF PAGES 13
16. DISTRIBUTION STATEMENT (of this Report) APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.		15. SECURITY CLASS. (of this report) Unclassified
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) censored data, exponential distribution, and sequential estimation		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The paper proposes a sequential estimator $\hat{\theta}$ of the parameter θ of an exponential distribution when the data is censored. Without any further conditions, it is shown that $\hat{\theta}$ is asymptotically risk efficient when the loss is measured by the squared error loss of estimation of $\hat{\theta}$ plus a linear cost function of the number of observations. In addition, it is shown that $\hat{\theta}$ is asymptotically normal as the cost per observation to zero.		